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Consequences of the Translation Invariance on the Darcy Free Convection Flow Past a Vertical Surface

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Abstract It is shown that the governing equation for the stream function of the Darcy free convection boundary layer flows past a vertical surface is invariant under arbitrary translations of the transverse coordinate y. The consequences of this basic symmetry property on the solutions corresponding to a prescribed surface temperature distribution $T_w(x)$ are investigated. It is found that starting with a "*primary solution*" which describes the temperature boundary layer on an *impermeable* surface, infinitely many "*translated solutions"* can be generated which form a continuous group, the "*translation group"* of the given primary solution. The elements of this group describe free convection boundary layer flows from *permeable* counterparts of the original surface with a transformed temperature distribution $\tilde{T}_w(x)$, when simultaneously a suitable lateral suction/injection of the fluid is applied. It turns out in this way that several exact solutions discovered during the latter few decades are in fact not basically new solutions, but *translated counterparts* of some formerly reported *primary solutions*. A few specific examples are discussed in detail.

Keywords Porous media · Free convection · Vertical surface · Temperature field · Translation invariance · Transpiration velocity

1 Introduction

The interest in fluid and heat flow in porous media stems from the widespread practical applications of these phenomena in modern industries and in many environmental issues as e.g. nuclear waste management, building thermal insulations, spread of pollutants, geothermal power plants, grain storage, packed-bed chemical reactors, oil recovery, ceramic processing, enhanced recovery of petroleum reservoirs, food science, medicine, etc. This circumstance has resulted in a vast amount of theoretical and experimental research work which has been collected and analyzed comprehensively in several recent works by Pop and Ingham (2001);

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Bejan et al. (2004), Ingham et al. (2004), Vafai (2005), and Nield and Bejan (2006), for example. In this large class of heat and fluid flow phenomena, the free convection past hot vertical surfaces embedded in saturated porous media has especially been promoted by important applications in geothermal power engineering (Cheng 1978). Following the seminal work of Cheng and Minkowycz (1977) and Cheng (1977) the theory of free convection boundary layer flows in porous media has experienced in the following decades an impetuous development, leading to a large number of exact and approximate analytical solutions and new knowledge (Merkin 1978, 1984; Minkowycz and Cheng 1982; Ingham and Brown 1986; Chaudhary et al. 1995; Magyari and Keller 2000; Magyari et al. 2002; Magyari and Rees 2006, 2008).

An overview of this vast literature emphasizes that with respect to the free convection flows past impermeable and permeable vertical surfaces two parallel, seemingly independent research fields have been aroused. This insight is the motivation of the present article. It is shown that the governing equation for the stream function of the Darcy free convection boundary layer flows is invariant under arbitrary translations of the transverse coordinate y. As a consequence of this basic symmetry property, it is found that starting with a "*primary solution*" which describes the temperature boundary layer on a *impermeable* surface, infinitely many "*translated solutions"* can be generated which form a continuous group, the "*translation group"* of the given primary solution. The elements of this group describe free convection boundary layer flows from *permeable* counterparts of the original surface with a transformed temperature distribution when simultaneously a suitable lateral suction/injection of the fluid is applied. It turns out in this way that several exact solutions discovered during the latter few decades are in fact not basically new solutions, but *translated counterparts* of some formerly reported *primary solutions*.

2 Basic Equations

Assuming that the boundary layer and the Boussinesq approximations hold, the governing continuity, Darcy and energy equations of the steady free convection flow over a semi-infinite vertical plate embedded in a fluid-saturated porous medium have the form (see e.g. Pop and Ingham 2001; Nield and Bejan 2006)

$$u_x + v_y = 0, \tag{1}$$

$$u = \frac{g\beta K}{\upsilon}\left(T - T_\infty\right), \tag{2}$$

$$u\,T_x + v\,T_y = \alpha_m\,T_{yy}, \tag{3}$$

In the above equations $x \geq 0$ and $y \geq 0$ are the Cartesian coordinates along and normal to the plate, respectively. The x axis points vertically upwards, u and v are the velocity components along x and y axes, T is the fluid temperature, T_∞ is the ambient temperature, K is the permeability of the porous medium, g is the acceleration due to gravity, c_p is the specific heat at constant pressure, α_m, β, and $\upsilon = \mu/\rho$ are the effective thermal diffusivity, thermal expansion coefficient, and kinematic viscosity, respectively, and the subscripts x and y indicate partial derivatives. It is further assumed that the plate in general is permeable and that its temperature distribution $T_w\,(x)$, as well as the distribution of transpiration velocity $v_w\,(x)$ are prescribed quantities, such that the boundary conditions read

$$\begin{array}{lll} T\,(x,0) = T_w\,(x)\,, & v\,(x,0) = v_w\,(x) & \text{on } y = 0 \\ T\,(x,y) \to T_\infty, & u\,(x,y) \to 0 & \text{as } y \to \infty \end{array} \tag{4}$$

On introducing the dimensionless coordinates X, Y, velocities U, V, temperature Θ, and stream function Ψ according to the definitions

$$\begin{aligned} X &= \frac{x}{L}, Y = Ra^{1/2}\frac{y}{L}, U = \frac{u}{u_0}, V = Ra^{1/2}\frac{v}{u_0}, \\ \Theta &= \frac{T - T_\infty}{T_*}, U = \Psi_Y, V = -\Psi_X \end{aligned} \tag{5}$$

the temperature and the mainstream velocity become coincident,

$$\Theta = U = \Psi_Y\,(X, Y) \tag{6}$$

and the energy equation goes over in

$$\Psi_Y\,(X, Y)\,\Psi_{XY}\,(X, Y) - \Psi_X\,(X, Y)\,\Psi_{YY}\,(X, Y) = \Psi_{YYY}\,(X, Y) \tag{7}$$

In Eq. 5, L denotes a reference length, $T_* > 0$ specifies the temperature scale, $u_0 = g\beta T_* K/\upsilon$ sets the scale of the mainstream velocity and Ra is the Darcy–Rayleigh number, $Ra = g\beta T_* KL/\left(\upsilon\alpha_m\right)$. The boundary conditions have now the form

$$\Psi_Y\,(X, 0) = \Theta_w\,(X)\,,\ \Psi_X\,(X, 0) = -V_w\,(X)\,,\ \Psi_Y\,(X, \infty) = 0 \tag{8}$$

where

$$\Theta_w\,(X) \equiv \Theta\,(X, 0) = \frac{T_w\,(x) - T_\infty}{T_*},\ V_w\,(X) \equiv V\,(X, 0) = Ra^{1/2}\frac{v_w\,(x)}{u_0} \tag{9}$$

3 The Translation Invariance

The boundary layer Eq. 7 which governs our free convection flow does not depend on the transverse coordinate Y explicitly. On this reason it is invariant under any uniform translation $Y \to Y + Y_0$ of this coordinate. Moreover, being founded on the intrinsic assumption that the thickness of the boundary layer is much smaller than the radius of curvature of the adjacent solid surface, the boundary layer approximation neglects the curvature terms of the governing equations. On this reason, Eq. 7 must be invariant not only with respect to uniform, but also with respect to any non-uniform translation

$$Y \to Y + Y_0\,(X) \tag{10}$$

of the transverse coordinate Y. This statement can be proved easily by the direct transformation

$$\Psi\,(X, Y) \to \Psi\,[X, Y + Y_0\,(X)] \equiv \tilde{\Psi}\,(X, Y) \tag{11}$$

of the stream function in Eq. 7 according to (10). Indeed, the transformation (10) leaves the Y-derivatives of $\tilde{\Psi}\,(X, Y)$ unchanged, while in the X-derivatives of $\Psi\,[X, Y + Y_0\,(X)]$ besides $\Psi_X\,[X, Y + Y_0\,(X)]$ the additional term $Y_0'\,(x)\,\Psi_Y\,[X, Y + Y_0\,(X)]$ occurs (everywhere in this article the prime denotes differentiation with respect to the argument). Thus, the right hand side of Eq. 7 is still $\tilde{\Psi}_{YYY}\,(X, Y)$, while its left hand side is transformed into

$$\begin{aligned} &\Psi_Y\,[X, Y + Y_0\,(X)]\left\{\Psi_{XY}\,[X, Y + Y_0\,(X)] + Y_0'\,(x)\,\Psi_{YY}\,[X, Y + Y_0\,(X)]\right\} \\ &\quad - \left\{\Psi_X\,[X, Y + Y_0\,(X)] + Y_0'\,(x)\,\Psi_Y\,[X, Y + Y_0\,(X)]\right\}\Psi_{YY}\,[X, Y + Y_0\,(X)] \end{aligned} \tag{12}$$

Now, in expression (12) the terms $\pm Y_0'(X)\,\Psi_Y[X, Y + Y_0(X)]\,\Psi_{YY}[X, Y + Y_0(X)]$ cancel each other so that Eq. 7 is transformed by the translation (10) into the equation

$$\tilde{\Psi}_Y(X, Y)\,\tilde{\Psi}_{XY}(X, Y) - \tilde{\Psi}_X(X, Y)\,\tilde{\Psi}_{YY}(X, Y) = \tilde{\Psi}_{YYY}(X, Y) \tag{13}$$

In this way it is proven that when $\Psi(X, Y)$ is a solution of Eq. 7, then $\tilde{\Psi}(X, Y) = \Psi[X, Y + Y_0(X)]$ also represents a solution of this equation. Consequently, the expressions of $\Theta = U = \Psi_Y$ and $V = -\Psi_X$ are transformed by Eqs. (10–11) as follows

$$\begin{aligned} &\Theta(X, Y) \to \tilde{\Theta}(X, Y) = \Theta[X, Y + Y_0(X)], \\ &V(X, Y) \to \tilde{V}(X, Y) = V[X, Y + Y_0(X)] - Y_0'(X)\,\tilde{\Theta}(X, Y) \end{aligned} \tag{14}$$

Accordingly, the boundary conditions (8) become

$$\begin{aligned} &\tilde{\Theta}(X, 0) \equiv \tilde{\Theta}_w(X) = \Theta[X, Y_0(X)], \\ &\tilde{V}(X, 0) \equiv \tilde{V}_w(X) = V[X, Y_0(X)] - Y_0'(X)\,\tilde{\Theta}_w(X) \end{aligned} \quad \text{at } Y = 0 \tag{15}$$

and

$$\tilde{\Theta}(X, Y) \to 0 \quad \text{as} \quad Y \to \infty \tag{16}$$

Therefore, the translation (10) of the transverse coordinate leaves the governing balance Eq. 7 as well as the asymptotic condition $\Psi_Y(X, \infty) = \Theta(X, \infty) = 0$ invariant, but it changes the wall conditions essentially for any nontrivial displacement function $Y_0(X)$. Accordingly, with the aid of the translations (10) one can generate physically new solutions from any specified solution $\{\Theta(X, Y), V(X, Y)\}$ of the problem (7–8). The new solutions $\left\{\tilde{\Theta}(X, Y), \tilde{V}(X, Y)\right\}$ given by Eq. 14, are associated with wall temperature and transpiration velocity distributions $\left\{\tilde{\Theta}_w(X), \tilde{V}_w(X)\right\}$ which, according to Eqs. (15) deviate in general from the initial ones $\{\Theta_w(X), V_w(X)\}$ substantially. It is worth mentioning here that, as an immediate consequence of Eqs. 15 and 16, the entrainment velocity $V(X, \infty) \equiv V_\infty$ is also invariant under the translations (10) of the transverse coordinate Y, $\tilde{V}_\infty = V_\infty$.

The solution corresponding to a given temperature distribution $\Theta_w(x)$ of an *impermeable* surface (i. e., to the case $V_w(x) \equiv 0$ in the second Eq. 8) will be referred to hereafter as *primary solution* of the boundary value problem (7–8). In the subsequent sections of this article, the effect of the translation invariance of Eq. 7 on the primary solutions describing various steady free convection boundary layer flows will be illustrated by several specific examples.

4 The Translation Group

The transformed stream functions $\tilde{\Psi}(X, Y) = \Psi[X, Y + Y_0(X)]$ generated from a given solution $\Psi = \Psi(X, Y)$ of Eq. 7 by displacements $Y_0 = Y_0(X)$ of the transverse coordinate Y are the elements of a continuous group, the *translation group*. Indeed, there exists a binary operation (composition rule) under which the elements of the set $\{\Psi[X, Y + Y_0(X)]\}$ are transformed in each other. This operation is the addition of two successive displacements $Y_{01}(X)$ and $Y_{02}(X)$ to a resulting one $Y_{03}(X) = Y_{01}(X) + Y_{02}(X)$. Thus, the composition rule is associative and commutative. The identity element corresponds to the zero translation, and the inverse element corresponds to the opposite displacement $-Y_0(X)$. Consequently, the set of solutions $\left\{\tilde{\Psi}(X, Y)\right\}$ of Eq. 7 generated by the translations (10) from some specified solution $\Psi(X, Y)$ forms a continuous group. This group will be referred to as *translation* *group* of the solution $\Psi(X, Y)$. Our main interest in the present article will be focused on the translation groups of *primary solutions*, i.e. of the solutions $\Psi(X, Y)$ corresponding to given temperature distributions $\Theta_w(x)$ of *impermeable* surfaces, $V_w(x) \equiv 0$.

5 Translations of the Self-Similar Solutions

5.1 The General Translations

As it is well known, Eq. 7 admits self-similar solutions of the form

$$\Psi(X, Y) = A(X)\, f(\eta), \eta = B(X)\, Y \tag{17}$$

when A and B are either power-law or exponential functions of the wall coordinate X. For the sake of simplicity, we consider in the present article only the case of power-law similarity

$$A(X) = A_0 X^{\frac{\lambda+1}{2}}, B(X) = B_0 X^{\frac{\lambda-1}{2}} \tag{18}$$

where A_0, B_0, and λ are dimensionless constants. For later convenience, we chose $A_0 = B_0 = \sqrt{a}$ where a is a positive (dimensionless) constant. In this way the components $U = \Theta = \Psi_Y$ and $V = -\Psi_X$ of the velocity field become

$$\begin{aligned} U(X, Y) &= \Theta(X, Y) = aX^{\lambda} f'(\eta), \eta = \sqrt{a} X^{\frac{\lambda-1}{2}} Y \\ V(X, Y) &= -\sqrt{a} X^{\frac{\lambda-1}{2}} \left[\frac{\lambda+1}{2} f(\eta) + \frac{\lambda-1}{2} \eta f'(\eta)\right] \end{aligned} \tag{19}$$

where the similar stream function $f = f(\eta)$ is obtained as solution of the two point boundary value problem

$$f''' + \frac{\lambda+1}{2} f f'' - \lambda f'^2 = 0, \tag{20}$$

$$f(0) = f_w, f'(0) = 1, f'(\infty) = 0 \tag{21}$$

The wall temperature and the transpiration velocity distributions are given by

$$\begin{aligned} \Theta(X, 0) &\equiv \Theta_w(X) = aX^{\lambda} \\ V(X, 0) &\equiv V_w(X) = -\sqrt{a} \frac{\lambda+1}{2} X^{\frac{\lambda-1}{2}} f_w \end{aligned} \tag{22}$$

where f_w denotes the mass transfer parameter (*dimensionless transpiration velocity*). The entrainment velocity in this case is

$$V(X, \infty) = V_\infty(X) = -\sqrt{a} \frac{\lambda+1}{2} X^{\frac{\lambda-1}{2}} f_\infty \tag{23}$$

where $f_\infty \equiv f(\infty)$ denotes the entrainment velocity parameter. The *primary solutions* are obtained now from Eqs. 19–22 for $f_w = 0$. As shown by Magyari et al. (2002), the case of inverse linear temperature distribution $\Theta_w(x) \sim 1/X$ corresponding to $\lambda = -1$, requires a special approach and will not further be considered here. We throughout assume in this article that $\lambda \neq -1$.

According to Eqs. 9 and 22, we have

$$T_w(x) = T_\infty + T_* \Theta_w(X) = T_\infty + aT_* X^{\lambda} \tag{24}$$

Thus, $a > 0$ implies that the wall temperature is everywhere larger than the ambient temperature, and the fluid motion is directed upwards.

Let us consider now a primary self-similar solution (19), i.e. a solution corresponding to the case $f_w = 0$ (impermeable surface in the boundary value problem (20–21)). According to Eq. 14 the transformed Eq. 19 is

$$\begin{aligned}&\tilde{\Theta}(X,Y) = \Theta_w(X)\, f'\left[\eta + \eta_0(X)\right],\\ &\tilde{V}(X,Y) = -\sqrt{a}X^{\frac{\lambda-1}{2}}\left[\tfrac{\lambda+1}{2} f\left[\eta+\eta_0(X)\right] + \tfrac{\lambda-1}{2}\left[\eta+\eta_0(x)\right] f'\left[\eta+\eta_0(X)\right]\right]\\ &\qquad -Y_0'(X)\,\tilde{\Theta}(X,Y)\end{aligned} \tag{25}$$

where

$$\eta_0(X) = \sqrt{a}X^{\frac{\lambda-1}{2}} Y_0(X) \tag{26}$$

is the displacement of the similarity variable η corresponding to the displacement $Y_0(X)$ of the transverse coordinate Y. With the aid of Eq. 26, Eq. 25 can also be transcribed in the form

$$\begin{aligned}&\tilde{\Theta}(X,Y) = \Theta_w(X)\, f'\left[\eta + \eta_0(X)\right],\\ &\tilde{V}(X,Y) = -\sqrt{a}X^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2} f\left[\eta+\eta_0(X)\right] + \frac{(\lambda-1)\eta + 2X\eta_0'(X)}{2} f'\left[\eta+\eta_0(X)\right]\right]\end{aligned} \tag{27}$$

The wall temperature and the transpiration velocity distributions which induce this transformed field are

$$\begin{aligned}\tilde{\Theta}_w(X) &= \Theta_w(X)\, f'\left[\eta_0(X)\right],\\ \tilde{V}_w(X) &= -\sqrt{a}X^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2} f\left[\eta_0(X)\right] + \frac{\lambda-1}{2}\eta_0(x)\, f'\left[\eta_0(X)\right]\right] - Y_0'(X)\,\tilde{\Theta}_w(X)\\ &= -\sqrt{a}X^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2} f\left[\eta_0(X)\right] + X\eta_0'(X)\, f'\left[\eta_0(X)\right]\right]\end{aligned} \tag{28}$$

Therefore, when the primary solution $f = f(\eta)$ of the original problem (20–21) is known and a displacement function $Y_0(X)$ is specified, Eqs. (25–27) give the corresponding transformed temperature and velocity fields associated with the wall temperature and transpiration velocity distributions (28). By this transformation the self similarity of the primary solution gets in general broken.

The first Eq. 28 shows that for a suitable choice of $\eta_0(X)$, i.e. of the displacement function $Y_0(X)$, any desired wall temperature distribution $\tilde{\Theta}_w(X)$ can be "prepared" from the initial distribution $\Theta_w(X)$ when the primary solution $f = f(\eta)$ is known. Indeed, denoting the inverse function of $f' = f'(\eta)$ by $f'^{-1}[\ldots]$, the first Eq. 28 yields for this "suitable" $\eta_0(X)$ the expression

$$\eta_0(X) = f'^{-1}\left[\frac{\tilde{\Theta}_w(X)}{\Theta_w(X)}\right] \tag{29}$$

Among the displacements (26) of the similarity variable η, the special case $\eta_0(x)$ =constant is of a basic physical and engineering interest and will be considered in some detail below.

5.2 Translations with $\eta_0(X)$ =Constant

In this case, the transformed fields (27) reduce to

$$\begin{aligned}\tilde{\Theta}(X,Y) &= \Theta_w(X)\, f'(\eta+\eta_0),\\ \tilde{V}(X,Y) &= -\sqrt{a}X^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2}f(\eta+\eta_0)+\frac{\lambda-1}{2}\eta f'(\eta+\eta_0)\right]\end{aligned} \tag{30}$$

Accordingly, the transformed wall functions (28) become

$$\tilde{\Theta}_w(X) = \Theta_w(X)\, f'(\eta_0),\ \tilde{V}_w(X) = -\sqrt{a}\frac{\lambda+1}{2}X^{\frac{\lambda-1}{2}}f(\eta_0) \tag{31}$$

The first Eq. 31 shows that for the special choice $\eta_0(X)$ = constant, the transformed wall temperature distribution $\tilde{\Theta}_w(X)$ is always proportional to $\Theta_w(X)$, in a full agreement with Eq. 29. According to Eq. 24, the constant of proportionality $f'(\eta_0)$ can be absorbed in the scale factor a, so that in this case the transformed wall temperature distribution $\tilde{\Theta}_w(X)$ is nothing more that a rescaled form of $\Theta_w(X)$. At the same time, the second Eq. 31 shows that the transformed counterpart $\left\{\tilde{\Theta},\tilde{V}\right\}$ of the primary solution $\{\Theta, V\}$ describes now the flow past a *permeable* surface with the transpiration velocity distribution $\tilde{V}_w(X)$. In the above sense, it is convenient to introduce the notations

$$\tilde{a} \equiv af'(\eta_0),\ \tilde{f}_w \equiv \frac{f(\eta_0)}{\sqrt{f'(\eta_0)}} \equiv H\, f(\eta_0),\ H \equiv \sqrt{\frac{a}{\tilde{a}}} \tag{32}$$

so that Eq. 31 become

$$\tilde{\Theta}_w(X) = \tilde{a}X^{\lambda},\ \tilde{V}_w(X) = -\sqrt{\tilde{a}}\frac{\lambda+1}{2}X^{\frac{\lambda-1}{2}}\tilde{f}_w \tag{33}$$

Consequently, Eq. 30 can be transcribed in the form

$$\begin{aligned}\tilde{\Theta}(X,Y) &= \tilde{\Theta}_w(X)\,\tilde{f}'(\tilde{\eta}),\\ \tilde{V}(X,Y) &= -\sqrt{\tilde{a}}X^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2}\tilde{f}(\tilde{\eta})+\frac{\lambda-1}{2}\tilde{\eta}\tilde{f}'(\tilde{\eta})\right]\end{aligned} \tag{34}$$

where

$$\tilde{f}'(\tilde{\eta}) \equiv \frac{f'(\eta+\eta_0)}{f'(\eta_0)},\ \tilde{f}(\tilde{\eta}) \equiv \frac{f(\eta+\eta_0)}{\sqrt{f'(\eta_0)}},\ \tilde{\eta} \equiv \frac{\eta}{H} = \sqrt{\tilde{a}}X^{\frac{\lambda-1}{2}}Y \tag{35}$$

Therefore, a translation with $\eta_0(X)$ =constant applied to a primary solution (impermeable surface, $f_w = 0$) generates a new solution which describes the flow corresponding to the physically equivalent power-law temperature distribution $\tilde{\Theta}_w(X) = \tilde{a}X^{\lambda}$ in the case when the surface is permeable and the transpiration velocity distribution $\tilde{V}_w(X)$ is effective. Equations 34 and 33 have the same form as Eqs. 19 and 22, respectively. This comparison shows that the solutions associated with non-vanishing values of the mass transfer parameter f_w do not represent basically new solutions of the boundary value problem (20–21), but are elements of the translation group of the primary solutions ($f_w = 0$) of this problem and, accordingly, can be generated from the latter ones by constant displacements of the similarity variable η. In this case, the self similarity is preserved. It is also worth mentioning here that, according to Eq. 26, the displacement $Y_0(X)$ of the transverse coordinate in general is not a constant when $\eta_0(X)$ =constant, except for the case $\lambda = 1$ where $\eta_0(X) = \sqrt{a}Y_0(X)$ holds.

In order to be more specific, in the following sections the translation groups of some exact analytical primary solutions ($f_w = 0$) of the boundary value problem (20–21) will be discussed in detail.

6 Translation Group of the Primary Solution with $\lambda = 1$

As it is well known, the boundary value problem (20–21) admits for the wall temperature distributions (24) *primary solutions* in a closed analytical form when $\lambda = 1$ and $\lambda = -1/3$ (see Ingham and Brown 1986). The case $\lambda = -1/3$ will be discussed in Sect. 7. The primary solution corresponding to the linear wall temperature distribution $\Theta_w(X) = aX$ has the simple form

$$
\begin{aligned}
&(\eta) = 1 - e^{-\eta},\ f'(\eta) = e^{-\eta},\ \eta = \sqrt{a}Y, \\
&\Theta(X,Y) = aXe^{-\eta},\ V(X,Y) = -\sqrt{a}\left(1 - e^{-\eta}\right), \\
&\Theta_w(X) = aX,\ V_w(X) = 0
\end{aligned}
\tag{36}
$$

The entrainment velocity parameter in this case is $f_\infty \equiv f(\infty) = 1$.

6.1 The General Translation Group

According to Eq. 27, the translation group of the primary solution (36) is given by

$$
\begin{aligned}
&\tilde{\Theta}(X,Y) = aXe^{-[\eta+\eta_0(X)]},\ \eta_0(X) = \sqrt{a}Y_0(X), \\
&\tilde{V}(X,Y) = -\sqrt{a}\left[1 - e^{-[\eta+\eta_0(X)]} + \eta_0'(X)\,Xe^{-[\eta+\eta_0(X)]}\right]
\end{aligned}
\tag{37}
$$

The transformed solution (37) corresponds to the wall functions

$$
\tilde{\Theta}_w(X) = aXe^{-\eta_0(X)},\ \tilde{V}_w(X) = -\sqrt{a}\left[1 - e^{-\eta_0(X)} + \eta_0'(X)\,Xe^{-\eta_0(X)}\right]
\tag{38}
$$

Equation 37 gives in terms of the displacement function $\eta_0(X) = \sqrt{a}Y_0(X)$ the translation group associated with the primary solution (36) in its most general form. In addition to this form, however, it is useful to express the transformed fields $\tilde{\Theta}(X,Y)$ and $\tilde{V}(X,Y)$ also in terms of the transformed wall temperature distribution $\tilde{\Theta}_w(X)$, instead of $\eta_0(X)$. This can be achieved with the aid of Eq. 29 which in the present case becomes

$$
\eta_0(X) = -\ln\left[\frac{\tilde{\Theta}_w(X)}{aX}\right]
\tag{39}
$$

Thus, expressed in terms of $\tilde{\Theta}_w(X)$, Eq. 37 read

$$
\begin{aligned}
&\tilde{\Theta}(X,Y) = \tilde{\Theta}_w(X)\,e^{-\eta}, \\
&\tilde{V}(X,Y) = -\sqrt{a}\left[1 - \frac{1}{a}\frac{d\tilde{\Theta}_w(X)}{dX}e^{-\eta}\right]
\end{aligned}
\tag{40}
$$

The corresponding transpiration velocity distribution is immediately obtained as

$$
\tilde{V}_w(X) = -\sqrt{a}\left[1 - \frac{1}{a}\frac{d\tilde{\Theta}_w(X)}{dX}\right]
\tag{41}
$$

This simple example emphasizes the astonishing consequences of the translation invariance clearly. Indeed, comparing the transformed solution (40) to the primary solution (36), the following features emerge.

1. The *same* self similar temperature boundary layer $e^{-\eta}$ can be generated not only by the linear wall temperature distribution $\Theta_w(X) = aX$ of an *impermeable* surface ($V_w(X) = 0$), but also by any other *arbitrary* wall temperature distribution $\tilde{\Theta}_w(X)$ when the surface is *permeable* and a lateral suction/injection of the fluid with a "suitable" velocity distribution $\tilde{V}_w(X)$ is applied.
2. The "suitable" transpiration velocity distribution $\tilde{V}_w(X)$ is obtained in terms of the wall temperature distribution $\tilde{\Theta}_w(X)$ by the simple relationship (41). In other words, the change from the linear wall temperature distribution $\Theta_w(X) = aX$ of an impermeable surface to an *arbitrary* temperature distribution $\tilde{\Theta}_w(X)$ of a permeable surface, can always be compensated by the effect of the transpiration velocity distribution (37), so that the similar temperature field $e^{-\eta}$ remains unchanged, i.e.

$$e^{-\eta} = \frac{\Theta(X,Y)}{\Theta_w(X)} = \frac{\tilde{\Theta}(X,Y)}{\tilde{\Theta}_w(X)} \tag{42}$$

We see, therefore, that the similar temperature field $e^{-\eta}$ is the same for the whole translation group of the primary solution (36), no matter the choice of $\tilde{\Theta}_w(X)$. The entrainment velocity $V(X,\infty) \equiv V_\infty$ is also the same for the whole translation group of the primary solution (39), $\tilde{V}_\infty = V_\infty = -\sqrt{a}$. To be more specific let us consider the example of a general polynomial wall temperature distribution

$$\tilde{\Theta}_w(X) = A_1X + A_2X^2 + \cdots + A_nX^n, \quad n = 2, 3, 4, \ldots \tag{43}$$

In this case, Eqs. 40 and 41 yield

$$\begin{aligned} \tilde{\Theta}(X,Y) &= \left(A_1X + A_2X^2 + \cdots + A_nX^n\right)e^{-\eta}, \eta = \sqrt{a}Y, \\ \tilde{V}(X,Y) &= -\sqrt{a} + \frac{1}{\sqrt{a}}\left(A_1 + 2A_2X + \cdots + nA_nX^{n-1}\right)e^{-\eta} \end{aligned} \tag{44}$$

and

$$\tilde{V}_w(X) = -\sqrt{a} + \frac{1}{\sqrt{a}}\left(A_1 + 2A_2X + \cdots + nA_nX^{n-1}\right) \tag{45}$$

respectively. For $A_1 = a$ and $A_2 = A_3 = \cdots = A_n = 0$, we recover in Eqs. 43–44 the primary solution (36). Now, bearing in mind that the primary solution corresponds to the linear wall temperature distribution $T_w(x) = T_\infty + T_*aX$, it is indeed surprising that Eqs. 43–45 which are associated with the wall temperature distribution

$$T_w(x) = T_\infty + T_*\left(A_1X + A_2X^2 + \cdots + A_nX^n\right) \tag{46}$$

with arbitrary $A_1, A_2, \ldots A_n$, also represent a solution of the free convection boundary value problem (7–8). Moreover, the solution (44) is only one element of the infinite translation group of the primary solution (36). Any other imaginable $\tilde{\Theta}_w(X)$ gives via Eqs. 40–41 a further solution.

6.2 Translations with $\eta_0(X) = \text{Constant}$

Bearing in mind the primary solution (36) for $f(\eta)$, the parameters (32) of the transformed solution obtained by a translation with $\eta_0(X) = \text{constant}$, as well as the corresponding wall functions (33) become

$$H = \sqrt{\frac{a}{\tilde{a}}} = e^{\frac{\eta_0}{2}}, \quad \tilde{f}_w = H - \frac{1}{H} \tag{47}$$

$$\tilde{\Theta}_w(X) = \tilde{a}X, \quad \tilde{V}_w(X) = -\sqrt{\tilde{a}}\,\tilde{f}_w \tag{48}$$

In this way, Eqs. 34 and 35 yield for the transformed solution the expressions

$$\tilde{\Theta}(X,Y) = \tilde{a}X\,e^{-H\tilde{\eta}}, \tilde{V}(X,Y) = -\sqrt{\tilde{a}}\left(\tilde{f}_w + \frac{1-e^{-H\,\tilde{\eta}}}{H}\right), \tilde{\eta} = \sqrt{\tilde{a}}Y \tag{49}$$

The second Eq. 47 gives the value of H in terms of the transformed transpiration parameter $\tilde{f}_w$ as

$$H = \frac{1}{2}\left(\tilde{f}_w + \sqrt{\tilde{f}_w^2 + 4}\right) \tag{50}$$

In Eqs. (48–50), we recover the exact solution reported by Magyari and Keller (2000) for the free convection boundary layer flow past a vertical permeable surface with the linear temperature distribution $\tilde{\Theta}_w(X) = \tilde{a}X$ and the uniform transpiration velocity $\tilde{V}_w(X) = -\sqrt{\tilde{a}}\,\tilde{f}_w$. We see in this way clearly that (49) is indeed not a basically new solution of the boundary value problem, but it is the transformed counterpart with $\eta_0(X) = \text{constant}$ of the primary solution of Ingham and Brown (1986).

7 Translation Group of the Primary Solution with $\lambda = -1/3$

The primary solution corresponding to the linear wall temperature distribution $\Theta_w(X) = aX^{-1/3}$ is (see Ingham and Brown 1986)

$$\begin{aligned}
&f(\eta) = \sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}}\right), f'(\eta) = 1 - \tanh^2\left(\frac{\eta}{\sqrt{6}}\right), \eta = \sqrt{a}X^{-2/3}Y,\\
&\Theta(X,Y) = \Theta_w(X)\left[1 - \tanh^2\left(\frac{\eta}{\sqrt{6}}\right)\right], \Theta_w(X) = aX^{-1/3},\\
&V(X,Y) = -\frac{\sqrt{a}}{3}X^{-2/3}\left\{\sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}}\right) - 2\eta\left[1 - \tanh^2\left(\frac{\eta}{\sqrt{6}}\right)\right]\right\}
\end{aligned} \tag{51}$$

The entrainment velocity parameter in this case is $f_\infty \equiv f(\infty) = \sqrt{6}$.

7.1 The General Translation Group

According to Eq. 27, the translation group of the primary solution (51) is given by

$$\begin{aligned}\tilde{\Theta}(X,Y) &= \Theta_w(X)\left[1-\tanh^2\left(\frac{\eta+\eta_0(X)}{\sqrt{6}}\right)\right],\\ \tilde{V}(X,Y) &= -\frac{\sqrt{a}}{3}X^{-2/3}\left\{\sqrt{6}\tanh\left(\frac{\eta+\eta_0(X)}{\sqrt{6}}\right)\right.\\ &\quad\left.-\left[2\eta-3X\eta_0'(X)\right]\left[1-\tanh^2\left(\frac{\eta+\eta_0(X)}{\sqrt{6}}\right)\right]\right\}\end{aligned} \tag{52}$$

The wall temperature and the transpiration velocity distributions which induce this transformed field are

$$\begin{aligned}\tilde{\Theta}_w(X) &= \Theta_w(X)\left[1-\tanh^2\left(\frac{\eta_0(X)}{\sqrt{6}}\right)\right],\\ \tilde{V}_w(X) &= -\frac{\sqrt{a}}{3}X^{-2/3}\left\{\sqrt{6}\tanh\left(\frac{\eta_0(X)}{\sqrt{6}}\right)+3X\eta_0'(X)\left[1-\tanh^2\left(\frac{\eta_0(X)}{\sqrt{6}}\right)\right]\right\}\end{aligned} \tag{53}$$

Equation 52 gives in terms of the displacement function $\eta_0(X)=\sqrt{a}X^{-2/3}Y_0(X)$ the translation group associated with the primary solution (51) in its most general form. We see again that for a suitable choice $\eta_0(X)$, i.e. of the displacement function $Y_0(X)$, any desired wall temperature distribution $\tilde{\Theta}_w(X)$ can be "prepared" from the initial distribution $\Theta_w(X)$. This "suitable" $\eta_0(X)$, which in the general case is given by Eq. 29, in the present case is

$$\eta_0(X)=\sqrt{6}\,\mathrm{arctanh}\left[\sqrt{1-\frac{\tilde{\Theta}_w(X)}{a\,X^{-1/3}}}\right] \tag{54}$$

Thus, in terms of $\tilde{\Theta}_w(X)$ the first Eq. 52 has the form

$$\tilde{\Theta}(X,Y)=\frac{\tilde{\Theta}_w(X)\left[1-\tanh^2\left(\frac{\eta}{\sqrt{6}}\right)\right]}{\left[1+\sqrt{1-\frac{\tilde{\Theta}_w(X)}{a\,X^{-1/3}}}\,\tanh\left(\frac{\eta}{\sqrt{6}}\right)\right]^2} \tag{55}$$

We see that in this case the self similarity gets broken, unless $\tilde{\Theta}_w(X)$ is proportional to $\Theta_w(X)$, i.e., $\eta_0(X)=$ constant.

7.2 The Case $\eta_0(x)=\text{const.}\equiv\eta_0$

In this case, Eq. 33 of the transformed wall functions reduce to

$$\tilde{\Theta}_w(X)=\tilde{a}X^{-1/3},\ \tilde{V}_w(X)=-\frac{\sqrt{\tilde{a}}}{3}X^{-2/3}\tilde{f}_w \tag{56}$$

where, according to Eq. 32 and the primary solution (51) we have

$$\tilde{f}_w=\frac{\sqrt{6}\tanh\left(\frac{\eta_0}{\sqrt{6}}\right)}{\sqrt{1-\tanh^2\left(\frac{\eta_0}{\sqrt{6}}\right)}} \tag{57}$$

Solved with respect to tanh $\left(\eta_0/\sqrt{6}\right)$, Eq. 57 gives

$$\tanh\left(\frac{\eta_0}{\sqrt{6}}\right) = \frac{\tilde{f}_w}{\tilde{f}_\infty},\ \tilde{f}_\infty \equiv \sqrt{6+\tilde{f}_w^2} \tag{58}$$

and thus Eq. 32 further yield

$$H = \sqrt{\frac{a}{\tilde{a}}} = \frac{\tilde{f}_\infty}{\sqrt{6}} \tag{59}$$

The transformed solution (34) reduces in this case to

$$\begin{aligned} \tilde{\Theta}(X,Y) &= \tilde{\Theta}_w(X)\,\tilde{f}'(\tilde{\eta}), \\ \tilde{V}(X,Y) &= -\frac{\sqrt{\tilde{a}}}{3}X^{-2/3}\left[\tilde{f}(\tilde{\eta}) - 2\tilde{\eta}\tilde{f}'(\tilde{\eta})\right] \end{aligned} \tag{60}$$

where, according to Eqs. 35, 51, 58, and 59

$$\begin{aligned} \tilde{f}(\tilde{\eta}) &= \tilde{f}_\infty \tanh\left(\frac{\tilde{f}_\infty}{6}\tilde{\eta} + \operatorname{arctanh}\frac{\tilde{f}_w}{\tilde{f}_\infty}\right),\ \operatorname{arctanh}\frac{\tilde{f}_w}{\tilde{f}_\infty} = \frac{1}{2}\ln\left(\frac{\tilde{f}_\infty+\tilde{f}_w}{\tilde{f}_\infty-\tilde{f}_w}\right), \\ \tilde{f}'(\tilde{\eta}) &= \frac{\tilde{f}_\infty^2}{6}\left[1-\tanh^2\left(\frac{\tilde{f}_\infty}{6}\tilde{\eta} + \operatorname{arctanh}\frac{\tilde{f}_w}{\tilde{f}_\infty}\right)\right],\ \tilde{\eta} = \sqrt{\tilde{a}}X^{-2/3}Y \end{aligned} \tag{61}$$

The first Eq. 61 shows that $\tilde{f}_\infty = \tilde{f}(\infty)$ represents the transformed entrainment velocity parameter. In Eqs. 56, 60, and 61, we recover the well-known exact solution of the free convection boundary value problem for the permeable surface when the wall temperature and the transpiration velocity distribution are given by Eq. 56 (see Magyari and Keller 2000; Magyari and Rees 2008).

8 Summary and Conclusions

The translation invariance of the boundary layer equation which governs the steady Darcy free convection past a vertical surface with prescribed temperature distribution $T_w(x)$, as well as the consequences of this basic symmetry property have been investigated. The main results of this article can be summarized as follows.

1. Staring with a given *primary solution* which describes the free convection flow past an impermeable surface, by the transformations $Y \to Y + Y_0(X)$ of the transverse coordinate infinitely many *translated solutions* can be generated.
2. The translated solutions form a continuous group, the *translation group* of the given primary solution.
3. The elements of the translation group describe free convection boundary layer flows from *permeable* counterparts of the original surface with a transformed temperature distribution $\tilde{T}_w(x)$ when simultaneously a lateral suction/injection $\tilde{V}_w(x)$ of the fluid is applied.
4. By a suitable choice of the displacement function $Y_0(X)$, from any given wall temperature distribution $T_w(x)$ and the corresponding solution, any other temperature distribution $\tilde{T}_w(x)$ can be prepared (e.g., from a linear temperature distribution, a general polynomial one; see Sect. 6.1). A solution corresponding to $\tilde{T}_w(x)$ can be obtained from the solution corresponding to $T_w(x)$ by a simple transformation, without needing to solve a new

boundary value problem separately. In other words, the change from a prescribed wall temperature distribution $T_w(x)$ to any other one $\tilde{T}_w(x)$ can always be influenced by a change from the transpiration velocity distribution $V_w(x)$ to a new one $\tilde{V}_w(x)$ in such a way that by these transformations a new solution of the boundary value problem results.

5. As a direct consequence of the above features, could be shown that some well-known exact solutions are in fact not basically new solutions, but *translated counterparts* of some formerly reported *primary solutions*.

It is worth mentioning here that the translation invariance has similar consequences also in the case of boundary layer flows induced in clear viscous fluids by continuous surfaces stretching or shrinking with prescribed velocities (Magyari 2010). The reason is the formal mathematical analogy between the governing equations for the stream function of these wall driven flows on the one hand, and the present case of the free convection boundary layer flows in fluid saturated porous media on the other hand. Obviously, the underlying physics and thus the physical consequences of the translation invariance in these two cases are basically different. The only common feature shared by these two types of boundary layer flows with respect to the translation invariance is the decisive effect of the wall transpiration, which makes possible that from every known solutions infinitely many new solutions can be generated.

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